

The Lorentz-Dirac and Landau-Lifshitz equations from the perspective of modern renormalization theory

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Abstract

In this pedagogical note, I revisit the problem of the equation of motion of a relativistic classical electron coupled to the electromagnetic field, a problem that is not so much addressed in the education of the typical physics student as put aside en route to the more difficult problems that arise in quantum field theory. The equations governing the motion of a classical electron under the influence of its electromagnetic field have been discussed for a century, and continue to be actively investigated in the current literature, but it appears that a consistent approach to the problem from the point of view of modern renormalization theory has not previously been reported. I show that the methods of modern renormalization theory applied to the full Maxwell-Lorentz system provide a natural and intuitive derivation of the Lorentz-Dirac equation as an effective description of the electron motion valid for distances large compared to the classical electron radius. Moreover, a consistent treatment of the Lorentz-Dirac equation as an effective theory leads directly to the Landau-Lifshitz equation and provides a clear conceptual basis for Rohrlich's recent approach to that equation (Phys. Rev. **E77**, 046609 (2008)). A full and careful computation of the self-field of the classical electron is given to provide the student a nontrivial problem in renormalization that does not require mastering the apparatus of quantum field theory.

I. INTRODUCTION

The theory of a classical, relativistic, charged particle interacting with the electromagnetic field is a venerable one. It is simple to formulate. As Coleman remarked in the article that is the principal inspiration for the present paper: “only the free electromagnetic field is simpler.”¹ That apparent simplicity is deceptive. The addition of a relativistic charged particle to Maxwell’s theory historically marks the first appearance of the “divergences” or “infinities” in theories of charged particles coupled to the electromagnetic field. Although pre-quantum theorists worried a great deal about such difficulties,² the quantum revolution overtook these worries. Theorists’ attention soon turned to quantum electrodynamics (QED), culminating in the Schwinger-Feynman-Dyson-Tomonaga formulation of renormalized QED.³ This refocusing of attention away from classical electrodynamics was vindicated by the detailed experimental successes of renormalized QED.⁴

That focus has remained unaltered ever since. Nearly twenty years after the formulation of renormalized QED, one author noted that “with very few exceptions the classical theory of charged particles has been largely ignored and has been left in an incomplete state since the discovery of quantum mechanics.”⁵ The relatively few occasions^{5,6} when the classical theory was revisited seemed mostly due to an underlying unease with the entire renormalization program.⁷ One can still find evidence of this unease in the current literature.⁸

Renormalization itself has undergone a significant conceptual transformation since Coleman’s application of renormalization methods to classical electron theory. Many different theoretical methods and problems fall under the rubric of “effective field theories”⁹ but the basic idea—the identification and separation of multiple physical length scales—is simple to grasp and independent of the complexities of quantum field theory. The purpose of this paper is to apply effective field theory ideas to classical electron theory to demonstrate how they simplify the conceptual aspects of the theory and organize systematically the computational labor involved in solving the theory. The tools required—dimensional analysis, some elements of classical relativistic field theory, delta function manipulations—should be within the grasp of advanced undergraduate and beginning graduate students.

II. SEPARATING OUT MULTIPLE SCALES IN THE MAXWELL-LORENTZ THEORY

The Maxwell-Lorentz theory of a relativistic classical charged particle interacting with the electromagnetic field is defined by the following coupled equations:¹⁰

$$m_0 \ddot{z}^\mu(\tau) = e_0 F^{\mu\nu}(z(\tau)) \dot{z}_\nu(\tau), \quad (1a)$$

$$\partial_\nu F^{\mu\nu}(x) = J^\mu(x), \quad (1b)$$

where the subsidiary definitions of the field strength and the current are:

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x), \quad (2a)$$

$$J^\mu(x) = e_0 \int_{-\infty}^{+\infty} \dot{z}^\mu(\tau') \delta^{(4)}(x - z(\tau')) d\tau'. \quad (2b)$$

To analyze these equations, it is convenient to specify the Lorenz family of gauges: $\partial_\lambda A^\lambda(x) = 0$. In this gauge, the equations of motion simplify to:

$$m_0 a^\mu(\tau) = e_0 F^{\mu\nu}(z(\tau)) v_\nu(\tau), \quad (3a)$$

$$-\partial^2 A^\mu(x) = J^\mu(x). \quad (3b)$$

We will use this coupled theory to compute the reaction of the radiation emitted by the electron on the electron's state of motion. The coupled equations of motion indicate the solution strategy: first solve Eq. (3b) and evaluate the self-field produced by the particle in a given state of motion, and then insert the self-field into Eq. (3a) and determine the radiation reaction on the particle's state of motion.

Two parameters, m_0 and e_0 , appear in the equations of motion. They have dimensions of mass and charge, respectively, but, until the theory has been analyzed and its physical consequences understood, they are to be viewed merely as parameters that enter into the theory.

Dimensional analysis yields valuable information. Keeping in mind that c (but not \hbar , which doesn't exist for us) is set to unity, the dimensions of the field potentials and strengths follow from Eq. (3b):

$$[A] = \frac{Q}{L}, \quad (4a)$$

$$[F] = [\partial A] = \frac{Q}{L^2}, \quad (4b)$$

where Q and L denote charge and length dimensions.

Inserting these dimensions into Eq. (3a) yields the relation $[m_0] = Q^2/L$. Although the relationships of the parameters e_0 and m_0 to the physical electron charge, e_{phys} , and inertial mass, m_{phys} , have yet to be determined, we can already see the emergence of a natural length scale in classical electrodynamics, namely, the classical electron radius, defined (up to a constant) by $r_c \sim e_{\text{phys}}^2/m_{\text{phys}}$.

Because the field strengths in Eq. (3a) must be evaluated on the particle worldline—a distance $0 \ll r_c$ from the electron—while the radiation field is required at distances $\gg r_c$, the dynamical Maxwell-Lorentz system encompasses an infinite range of length scales. Therefore, the Maxwell field is a mixture of both near-field ($\ll r_c$) and far-field ($\gg r_c$) components. As Wilson¹¹ was perhaps the first to emphasize, it is the simultaneous presence of many coupled degrees of freedom that underlies the difficulty in solving multiscale problems. In the case of classical electron theory, it is the important and intimate coupling of the near-field component to the electron all the way to zero distance that has caused a century or more of confusion and difficulty.

Renormalization theory provides a systematic method for dealing with such multiscale problems. It is a three-step process. First, one introduces a finite length scale, ϵ . This cutoff explicitly parametrizes the theory’s sensitivity to short-distance physics. It defines the shortest distance that will be considered in the theory—physical degrees of freedom characterized by length scales smaller than the cutoff are excluded from the theory. In the limit that the $\epsilon \rightarrow 0$, all length scales are included. The cutoff is itself arbitrary. Once the cutoff is in place, all computed quantities are finite (and cutoff-dependent). The theory is then said to be “regularized.”

The second step of the renormalization process is to systematically disentangle the different length scales in the quantity of interest. A suitable Taylor expansion of the quantity of interest is often a convenient way to separate out short-distance (i.e., cutoff-dependent) behavior. For example, in quantum field theory, a Taylor expansion of an amplitude about some convenient point will separate the amplitude’s high-momentum (i.e., cutoff-dependent) behavior from its low-momentum (i.e., cutoff-independent) behavior. In the present case, the essential quantity of interest is the electron’s self-field. The expansion in powers of the cutoff is a perturbative expansion in the ratio of the short-distance scale (the cutoff) to the long-distance scales that characterize the motion of the electron and the radiation field.¹²

If the cutoff procedure respects the underlying symmetries of the theory, the terms in the expansion of the self-field are determined (up to multiplicative constants) by those symmetries and simple dimensional analysis. Truncating the resulting expansion of the self-field will be valid if, and only if, the motions of the electron are such that higher-order terms in the self-field expansion may be consistently and legitimately neglected. This seemingly innocent point will prove to be of great importance in what follows.

The third and final step of the renormalization process is to relate the “bare” parameters m_0 and e_0 to the physical mass and charge of the electron, m_{phys} and e_{phys} . The latter are defined according to some long-distance prescription: for example, m_{phys} is the inertia of a slowly moving electron in the presence of a weak external field, and e_{phys} is the coefficient of the Coulomb potential of a stationary charge at infinite distance away from the charge. The specific form of these relationships between the bare and physical parameters will depend on both the value of the cutoff parameter and the method by which the theory is regularized. Multiscale physical theories have the peculiar feature that we have to solve the theory (at least approximately) at all scales before we can normalize the theory’s parameters to long-distance measurements. This fact makes normalizing the theory a nontrivial endeavor, and one can view renormalization theory as a systematic way to accomplish that task. At any rate, normalizing the theory’s parameters completes the separation of short-distance from long-distance behavior in the theory. The equations that are left are long-distance equations, with long-distance parameters, valid at distances much greater than the cutoff.

A. The self-field of the classical electron

Armed with the finite cutoff ϵ , consider again Eq. (3a). The self-field $F_\epsilon^{\mu\nu}(z)$ —where the dependence on the cutoff is explicitly indicated—that appears in this equation contains both short-distance (cutoff-dependent) and long-distance (cutoff-independent) contributions. To implement the renormalization program, we need to systematically separate those two contributions.

The dependence of $F_\epsilon^{\mu\nu}(z(\tau))$ on the particle coordinates is largely determined by symmetry and dimensional considerations. First, $F_\epsilon^{\mu\nu}$ must have the structure of an antisymmetric Lorentz tensor. Second, it can only depend on the long-distance variables $v^\mu = \dot{z}^\mu$, $a^\mu = \dot{v}^\mu$, $\dot{a}^\mu = \ddot{v}^\mu$, $\ddot{a}^\mu = \dddot{v}^\mu$, etc., in a local fashion (i.e., all the particle variables

must be taken at the same proper time). Finally, its dimension must be $[F] = Q/L^2$.

For example, consider a term contributing to $F_\epsilon^{\mu\nu}(z(\tau))$ such as $e_0(v^\mu a^\nu - v^\nu a^\mu)$. Because $[v] = 1$, $[a] = L^{-1}$, and, in general, $[d^n v/d\tau^n] = L^{-n}$, a compensating factor with a length dimension unity must be inserted to make the contribution to F dimensionally correct. Because the current that appears in Eq. (2b) has no intrinsic length (it refers to a point particle, after all), the only available factor to use is the cutoff ϵ , yielding:

$$F_\epsilon^{\mu\nu}(z) \sim e_0 \frac{(v^\mu a^\nu - v^\nu a^\mu)}{\epsilon}. \quad (5)$$

This contribution is $O(\epsilon^{-1})$ and is therefore sensitive to near-field physics. The generality of this derivation—which depends on little more than locality, dimensional analysis, and Lorentz symmetry—makes it clear that such sensitivity to short-distance physics is an intrinsic part of the Maxwell-Lorentz system and is not an artifact of any particular solution method.

As another example, consider a term such as $(d^5 v^\mu/d\tau^5)(d^8 v^\nu/d\tau^8)$. Antisymmetrizing and inserting the correct number of ϵ s yields the contribution:

$$F_\epsilon^{\mu\nu}(z) \sim e_0 \epsilon^{11} \left(\frac{d^5 v^\mu}{d\tau^5} \frac{d^8 v^\nu}{d\tau^8} - \frac{d^5 v^\nu}{d\tau^5} \frac{d^8 v^\mu}{d\tau^8} \right). \quad (6)$$

This $O(\epsilon^{11})$ contribution is extremely insensitive to near-field physics. (Except for extremely violent accelerations, of which more later.)

These two examples point to a method of organizing the possible contributions to the self-field: simply write down all possible local combinations of the velocity and its derivatives, antisymmetrize appropriately, count powers of the length dimension, and then add in the requisite number of factors of ϵ to make the self-field dimensionally correct. The terms with negative powers of ϵ will be quite sensitive to near-field physics, those with positive powers of ϵ will be quite insensitive to near-field physics, and terms that do not depend on ϵ , $O(1)$ terms, will be marginally sensitive to near-field physics.

Using this procedure, we can easily write down the lower-order contributions to the self-field. An $O(\epsilon^{-2})$ term would be generated by a product such as $v^\mu v^\nu$, but this product vanishes upon antisymmetrization. The only $O(\epsilon^{-1})$ term is given by Eq. (5), because that is the sole term with a single power of $\dot{v} = a$ that has the appropriate Lorentz index structure. $O(1)$ terms could be generated in two possible ways: one power of $\ddot{v} = \dot{a}$ together with a v or two powers of $\dot{v} = a$. Because $a^\mu a^\nu$ vanishes upon antisymmetrization, the only remaining

$O(1)$ term is:

$$F_\epsilon^{\mu\nu}(z) \sim e_0 (v^\mu \dot{a}^\nu - v^\nu \dot{a}^\mu). \quad (7)$$

$O(\epsilon)$ contributions can come in three ways: one power of $\ddot{v} = \ddot{a}$ with a v ; one power each of \ddot{v} and \dot{v} ; or three powers of \dot{v} , appropriately contracted with a v to get the right number of Lorentz indices. The latter case is of the form $\dot{v}^\mu \dot{v}^\nu \dot{v}^\lambda v_\lambda$, which vanishes in two ways: either because of antisymmetrization between the μ and ν indices, or because of the identity $\dot{v}^\lambda v_\lambda = 0$. The remaining contributions at $O(\epsilon)$, then, are:

$$F_\epsilon^{\mu\nu}(z) \sim e_0 \epsilon (v^\mu \ddot{a}^\nu - v^\nu \ddot{a}^\mu), \quad (8a)$$

$$F_\epsilon^{\mu\nu}(z) \sim e_0 \epsilon (a^\mu \dot{a}^\nu - a^\nu \dot{a}^\mu). \quad (8b)$$

$$F_\epsilon^{\mu\nu}(z) \sim e_0 \epsilon (v \cdot \dot{a}) (v^\mu a^\nu - v^\nu a^\mu), \quad (8c)$$

where the latter two equations show the different non-vanishing ways in which one power each of $\ddot{v} = \ddot{a}$ and $\dot{v} = \dot{a}$ can be contracted (with extra v 's, if need be) to get the correct Lorentz structure.¹³

As we go to higher powers of ϵ , we can repeat this procedure, at the cost of more and more terms coming in. The end result of these simple considerations is that, up to terms of $O(\epsilon^2)$ and higher, the regularized self-field must have the long-distance expansion:

$$e_0^{-1} F_\epsilon^{\mu\nu}(z) = c_{-1} \epsilon^{-1} (v^\mu a^\nu - v^\nu a^\mu) \quad (9a)$$

$$+ c_0 (v^\mu \dot{a}^\nu - v^\nu \dot{a}^\mu) \quad (9b)$$

$$+ \epsilon \left[c_1^{(1)} (v^\mu \ddot{a}^\nu - v^\nu \ddot{a}^\mu) + c_1^{(2)} (a^\mu \dot{a}^\nu - a^\nu \dot{a}^\mu) + c_1^{(3)} (v \cdot \dot{a}) (v^\mu a^\nu - v^\nu a^\mu) \right] \quad (9c)$$

$$+ O(\epsilon^2), \quad (9d)$$

where $\{c_{-1}, c_0, c_1^{(1,2,3)}\}$ are undetermined constants, independent of $z(\tau)$ and ϵ .

In this expansion, ϵ is a finite parameter. For the expansion of the self-field in effective interactions to remain sensible to any given order in ϵ , say n , the accelerations suffered by the particle must not be so violent that the neglected terms of $O(\epsilon^{n+1})$ become comparable to the retained terms. To ensure that the expansion does not break down, the derivatives of the particle motion must be restricted by the condition $\epsilon d/d\tau \ll 1$, or $d/d\tau \ll \epsilon^{-1}$, i.e., the proper time interval, $\Delta\tau$, over which a significant change in the electron's motion can occur must satisfy the constraint $\Delta\tau \gg \epsilon$. We may consistently neglect higher order terms in the self-field if, and only if, we restrict our attention to electron motions that do not

change significantly over scales comparable to the cutoff. That is to say, the requirement of the cutoff theory that no short-distance degrees of freedom are explicitly present must be extended consistently to the electron motions that we admit into the theory.

B. Normalizing the theory at long distances

Let us assume that we are considering electron motions that only vary significantly over scales $\gg \epsilon$. Then we may consistently neglect all terms of $O(\epsilon)$ and above in the effective interaction expansion of the self-field. To this order, then, insert the self-field from Eq. (9a) into the particle equation of motion, Eq. (3a). Using the identities $v^2 = -1$, $v \cdot a = 0$ and $\dot{a} \cdot v = -a^2$, there results:

$$m_0 a^\mu(\tau) = F_\epsilon^{\mu\nu}(z) v_\nu = e_0^2 [c_{-1} \epsilon^{-1} a^\mu + c_0 (\dot{a}^\mu - a^2 v^\mu)] + O(\epsilon). \quad (10)$$

A slight rearrangement yields:

$$(m_0 - e_0^2 c_{-1} \epsilon^{-1}) a^\mu(\tau) = e_0^2 c_0 (\dot{a}^\mu - a^2 v^\mu) + O(\epsilon). \quad (11)$$

It is convenient for present purposes to add in an external field, giving the final equation of motion:

$$(m_0 - e_0^2 c_{-1} \epsilon^{-1}) a^\mu(\tau) = e_0 F_{\text{ext}}^{\mu\nu} v_\nu + e_0^2 c_0 (\dot{a}^\mu - a^2 v^\mu) + O(\epsilon). \quad (12)$$

We now have to address the question, long postponed, of the physical interpretation of the constants m_0 and e_0 . Let us presume, for the moment, that we have an interpretation of e_0 , and focus on m_0 . Consider a slow, very slightly accelerated motion. For such motions, the self-field contribution may be neglected, and Eq. (13) reduces to:

$$(m_0 - e_0^2 c_{-1} \epsilon^{-1}) a^\mu(\tau) = e_0 F_{\text{ext}}^{\mu\nu} v_\nu. \quad (13)$$

The coefficient multiplying the acceleration clearly functions as the electron inertia in the context of this theory. Therefore, the parameter m_0 must be chosen (“tuned”) so that the electron inertia term is the physical mass of the electron:

$$m_0 \equiv m_{\text{phys}} + e_0^2 c_{-1} \epsilon^{-1}. \quad (14)$$

The dynamics of the coupled field-particle system renders the normalization of m_0 nontrivial in the sense that short-distance physics (signaled by the presence of ϵ) enters into the

relationship between the parameter m_0 and the physical quantity m_{phys} . Before the advent of effective field theory ideas, this process was typically characterized as saying that the electron mass was “renormalized” by the self-field of the electron. But in the current approach, it is much more natural to say that the normalization of the mass parameter in Maxwell-Lorentz electrodynamics incorporates, or is perhaps dominated by, short-distance contributions to the dynamics of the system.

What about e_0 ? To normalize this parameter, consider the very simple case of an electron at rest at the origin. Writing the potential as $A^\mu = (\Phi, 0)$, Maxwell’s equations, Eq. (3b), simplify to

$$-\nabla^2\Phi(\mathbf{x}) = e_0\delta^{(3)}(\mathbf{x}), \quad (15)$$

which implies that the normalization of the charge parameter e_0 remains trivial ($e_0 = e_{\text{phys}}$) even in the coupled theory. Although often overlooked, this “no-renormalization” result is actually a deep (and experimentally incorrect) implication of classical electrodynamics. In quantum electrodynamics, closed electron loops render the normalization of the electron charge highly nontrivial indeed, and the analogous physical processes in quantum chromodynamics lead to the phenomenon of asymptotic freedom.

We have managed to extract a great deal of physical information out of the Maxwell-Lorentz theory while doing hardly any computations at all. The dependence of the mass normalization on short-distance physics has been brought out, and the form of the reaction force on the radiating electron has been arrived at quite simply. Moreover, the region of expected validity of this form emerges naturally. Both computational ease and the presence of undetermined constants are typical aspects of effective field theory methods. The price to be paid is that there remain two undetermined constants, c_{-1} and c_0 . The constant c_{-1} has been effectively submerged in the mass normalization. It remains to compute c_0 from the underlying theory.

The appendices do just that in considerable detail, with the following results. First, the value of c_{-1} depends on the method of regularization chosen. For the regulator used in the sequel, its value is:

$$c_{-1} = -\frac{1}{2} \left(\frac{1}{4\pi} \right). \quad (16)$$

Second, the value of c_0 is independent of the method of regularization and is:¹⁴

$$c_0 = \frac{2}{3} \left(\frac{1}{4\pi} \right). \quad (17)$$

The end result of the normalization process is the standard Lorentz-Dirac equation:

$$m_{\text{phys}} a^\mu(\tau) = e_{\text{phys}} F_{\text{ext}}^{\mu\nu} v_\nu + \frac{2}{3} \left(\frac{e_{\text{phys}}^2}{4\pi} \right) (\dot{a}^\mu - (a^\lambda a_\lambda) v^\mu) + O(\epsilon), \quad (18)$$

where the “ $O(\epsilon)$ ” is there to remind us that we are dropping higher order terms—dubbed “structure” terms in the older literature—in the effective interaction expansion of the self-field. In particular, we should always remember that the Lorentz-Dirac equation applies only to electron motions that change significantly on spatio-temporal scales $\gg \epsilon$. For electron motions that do change significantly over scales $\sim \epsilon$, the structure terms are no longer negligible, and the Lorentz-Dirac equation is no longer valid.

Coleman was well aware of the possibility that sufficiently violent electron motions could overwhelm any finite power of ϵ and ensure that the structure-dependent terms would dwarf the structure-independent terms. For that reason, he argued that it was necessary to take the limit $\epsilon \rightarrow 0$.

The arguments of modern renormalization theory are diametrically opposed to this point of view. We acknowledge that Eq. (14) implies that the limit $\epsilon \rightarrow 0$ is ill-defined. Even though the theory fails at distances $\ll \epsilon$, we can still use the theory to describe physics at distances $\gg \epsilon$. Rather than send the cutoff to zero, we keep it finite and restrict the possible electron motions we can include in the effective long-distance theory to those that will not invalidate the neglect of the structure-dependent terms in the effective theory. As we will see in the next section, this restriction is essential to the success of the effective approach in general and will have important ramifications for the interpretation and use of the Lorentz-Dirac equation.

III. ENFORCING CONSISTENCY AND ELIMINATING RUNAWAYS

Because the cutoff ϵ is arbitrary, we can set it at any convenient point. A natural value is the classical electron radius $\epsilon \sim r_c \equiv e_{\text{phys}}^2 (4\pi)^{-1} m_{\text{phys}}^{-1}$. It is conventional in the literature to modify this choice slightly by defining the scale $\tau_0 \equiv 2r_c/3$. With this choice, we can rewrite Eq. (18) in an interesting way:

$$a^\mu(\tau) = f^\mu(\tau) + \tau_0 (\dot{a}^\mu - (a^\lambda a_\lambda) v^\mu) + O(\tau_0^2), \quad (19)$$

where $f^\mu \equiv e_{\text{phys}} F_{\text{ext}}^{\mu\nu} v_\nu / m_{\text{phys}}$.¹⁵ As our derivation has shown, this equation is only valid for relatively gentle accelerations that occur on scales $\gg \tau_0$.

Eq. (19) presents some puzzling questions that date back to the earliest investigations in this area. The source of the trouble is the \dot{a} term that emerged in the self-field expansion above. Taken at face value, it would appear that treating Eq. (19) as an initial-value problem would require the initial specification of the acceleration of the electron, in addition to the usual position and velocity. Even worse, consider the simple case of a low-velocity electron with a vanishing external field. Eq. (19) reduces to

$$a^\mu(\tau) = \tau_0 \dot{a}^\mu(\tau), \quad (20)$$

which has an exponentially increasing solution, $a^\mu(\tau) = a^\mu(0)e^{\tau/\tau_0}$ in addition to the expected solution $a^\mu = 0$ (i.e., $v^\mu = \text{const}$). These additional solutions are aptly dubbed “runaways.” Had we chosen to work to a higher order in τ_0 , even higher derivatives would have appeared in the equations, necessitating the initial specification of the third, fourth, ..., derivatives of the electron position, presumably with additional runaway solutions. It seems, then, that our strategy of separating out the short-distance physics has fundamentally changed the nature of the dynamical system, and has apparently permitted some questionable solutions to intrude.

It is immediately clear that the runaway solution violates the basic restriction on the self-field expansion, namely, that $\tau_0 \dot{a}^\mu \ll a^\mu$. We are faced with the problem of constraining the long-distance effective theory in such a way that the solutions of the long-distance theory do not violate the assumptions under which the theory was derived. In other words, runaways have to be excluded.

In a very brief note,¹⁶ Bhabha pointed out that the fundamental distinction between the runaway solutions and the usual solutions of Eq. (19) lies in the fact that the runaways are not analytic functions of τ_0 near $\tau_0 = 0$. He proposed discarding all solutions that are not analytic in τ_0 as unphysical. Bhabha’s criterion turns out to be intimately related to the restriction we found necessary to impose on the possible electron motions to preserve the validity of the effective interaction expansion. As we have emphasized, modern renormalization methods solve multiscale problems by separating out the short- and long-distance scales, and then writing an effective long-distance equation, which in general involves higher-order derivatives in the long-distance degrees of freedom. For this expansion to remain under control, the higher-order terms in τ_0 must remain small compared to the lower-order terms in τ_0 . Therefore, the appropriate solutions of the long-distance equations must be Taylor-

expandable in terms of the short-distance cutoff, which is just Bhabha's criterion.

Bhabha's insight leads naturally to the idea that simple perturbation theory could eliminate runaway solutions to the Lorentz-Dirac equation. His basic idea has been generalized to a wide variety of effective theories generated in such diverse contexts as quantum field theory, string theory, and general relativity. All these effective long-distance theories involve high-order derivatives and runaway solutions, and require some sort of constraint to eliminate the unphysical solutions.¹⁷

In the current context, the method of perturbative constraints is straightforward to implement. Following Rohrlich,¹⁸ use the identity $a^2 = -v \cdot \dot{a}$ to rewrite Eq. (19) as:

$$a^\mu(\tau) = f^\mu(\tau) + \tau_0 (g^{\mu\nu} + v^\mu v^\nu) \dot{a}_\nu + O(\tau_0^2), \quad (21a)$$

$$= f^\mu(\tau) + \tau_0 \Pi^{\mu\nu} \dot{a}_\nu + O(\tau_0^2), \quad (21b)$$

where $\Pi^{\mu\nu}$ is a projector that eliminates any component parallel to the electron velocity. We now seek a perturbative solution to Eq. (21b) in the small parameter τ_0 : $a^\mu = a_0^\mu + \tau_0 a_1^\mu + O(\tau_0^2)$. Inserting this expansion into Eq. (21b), and matching powers of τ_0 , we find the $O(1)$ equation is simply $a_0^\mu = f^\mu$ and the $O(\tau_0)$ equation is:

$$a_1^\mu = \Pi_0^{\mu\nu} \dot{a}_0_\nu = \Pi_0^{\mu\nu} \dot{f}_\nu, \quad (22)$$

where we have inserted the result of the $O(1)$ equation. The perturbatively constrained solution to the Lorentz-Dirac equation is then:

$$a^\mu = f^\mu + \tau_0 \Pi^{\mu\nu} \dot{f}_\nu = f^\mu + \tau_0 (g^{\mu\nu} + v^\mu v^\nu) \dot{f}_\nu + O(\tau_0^2), \quad (23)$$

where we have replaced $\Pi_0^{\mu\nu}$ with $\Pi^{\mu\nu}$, which is correct to the order of approximation indicated. Note that the enforcement of the perturbative constraint has eliminated the \dot{a} term. If we consider the case of vanishing external force, Eq. (23) simplifies to $a^\mu = 0$, with solution $v^\mu = \text{const}$, we see that the runaway solution has been eliminated as well. Finally, we see that Rohrlich's criterion, $\tau_0 \dot{f}^\mu \ll f^\mu$, is simply a restatement of the basic criterion of validity of the Lorentz-Dirac equation when regarded as a long-distance effective approximation to the Maxwell-Lorentz theory.

IV. CONCLUSIONS

Rohrlich¹⁸ calls Eq. (23) the “physical Lorentz-Abraham-Dirac” equation. It is also often called the “Landau-Lifshitz” equation,²⁰ discussions of which are usually accompanied by various worries about its relationship to the Lorentz-Dirac equation. In my view, these worries are groundless. The thrust of the argument in this paper is that these equations follow directly from the consistent application of modern renormalization theory to the Maxwell-Lorentz theory, now viewed as an effective rather than a fundamental theory. The Lorentz-Dirac equation emerges as a long-distance approximation within the Maxwell-Lorentz theory that, when constrained in a manner appropriate to the expansion in derivatives characteristic of effective theories, simply and directly yields the Landau-Lifshitz equation. It bears repeating that Rohrlich’s criterion for the validity of the Landau-Lifshitz equation—which he postulated as the physical basis for Spohn’s¹⁹ mathematical reduction—is simply the criterion of validity for truncating the long-distance effective expansion of the electron self-field. Indeed, the entire thread of the argument in this paper can be viewed as a systematic reinterpretation of Rohrlich’s classic treatise in light of the advent of the effective field theoretic concepts and techniques that have come to dominate the modern interpretation of quantum field theory in the past two decades.

Appendix A: Regularization and explicit computation of the self-field

There is no substitute for explicit computation. To compute the coefficients in the effective interaction expansion used to analyze the long-distance behavior of the electron we will integrate Eq. (3b) using the retarded Green function:⁵

$$D_R(x) = \frac{1}{2\pi} \theta(x^0) \delta(x^2). \quad (\text{A1})$$

Up to a free field (the “in” field), which is taken to vanish,²¹ the solution for the retarded potential at a general spacetime point x is:

$$A_R^\mu(x) = e \int_{-\infty}^{\infty} d\tau D_R(x - z(\tau)) v^\mu(\tau) = \frac{e}{2\pi} \int_{-\infty}^0 d\tau \delta((x - z(\tau))^2) v^\mu(\tau), \quad (\text{A2})$$

where the θ function constraint has been taken into account in the latter term, and we have chosen to define the origin of proper time by $z^0(\tau)|_{\tau=0} = x^0$. (We have put $e_{\text{phys}} = e$ for

simplicity.) As long as the point x is off the particle worldline, this formula is well-defined and yields the usual Liénard-Wiechert potentials.

What is needed in Eq. (3a) is the value of the field at a point on the worldline, say $x = z(0)$. In that case, Eq. (A2) diverges and requires regularization. The regulator we will use is simple and convenient and involves replacing Eq.(A2) by:²²

$$A_\epsilon^\mu(x) = \frac{e}{2\pi} \int_{-\infty}^0 d\tau \delta((x - z(\tau))^2 + \epsilon^2) v^\mu(\tau). \quad (\text{A3})$$

Although we wish to ultimately compute the field strength, a useful warmup exercise is to compute the retarded potential on the electron worldline. In Eq. (A3), expand the argument of the delta function near $\tau = 0$ using:

$$z(\pm\tau) = z(0) \pm v(0)\tau + a(0)\tau^2/2 + O(\tau^3), \quad (\text{A4a})$$

$$(z(\pm\tau) - z(0))^2 = v(0)^2\tau^2 + O(\tau^3) = -\tau^2 + O(\tau^3). \quad (\text{A4b})$$

The delta function then becomes (recall $x = z(0)$):

$$\delta(-\tau^2 + \epsilon^2) = \delta(\tau^2 - \epsilon^2) = \frac{1}{2\epsilon} (\delta(\tau - \epsilon) + \delta(\tau + \epsilon)), \quad (\text{A5})$$

and we can immediately carry out the integration (only the second delta function contributes) to obtain:

$$A_\epsilon^\mu(z(0)) = \frac{e}{4\pi} \frac{v^\mu(-\epsilon)}{\epsilon}. \quad (\text{A6})$$

Expanding $v(-\epsilon) = v(0) - \epsilon a(0) + O(\epsilon^2)$, we see both a divergent and a finite term emerge:

$$A_\epsilon^\mu(z(0)) = \frac{e}{4\pi} \left(\frac{v^\mu(0)}{\epsilon} - a^\mu(0) \right) + O(\epsilon), \quad (\text{A7})$$

which can be immediately generalized to any point on the particle worldline:

$$A_\epsilon^\mu(z(\tau)) = \frac{e}{4\pi} \left(\frac{v^\mu(\tau)}{\epsilon} - a^\mu(\tau) \right) + O(\epsilon), \quad (\text{A8})$$

Note that the divergent term has the structure of a relativistic Coulomb potential and is present even for an unaccelerated motion, in complete accord with our intuitive notions of the short-distance coupling of the near-field with the electron. The simultaneous presence of divergent and nondivergent terms in Eq. (A8) is a clear sign of the multiscale nature of the coupled problem.

The computation of the field is more involved. It is convenient to take a derivative with respect to x before computing the integral:

$$\partial^\lambda A_\epsilon^\mu(x) = \frac{e}{2\pi} \int_{-\infty}^0 d\tau \delta'((x - z - \tau))^2 + \epsilon^2) 2(x - z(\tau))^\lambda v^\mu(\tau). \quad (\text{A9})$$

Now we insert a very useful identity from Dirac's 1938 paper:

$$\delta'((x - z(\tau))^2) = -\frac{1}{2} \frac{1}{(x - z(\tau)) \cdot v(\tau)} \frac{d}{d\tau} \delta((x - z(\tau))^2), \quad (\text{A10})$$

into the equation for the field strength, and integrate by parts to obtain the compact form:

$$\partial^\lambda A_\epsilon^\mu(x) = \frac{e}{2\pi} \int_{-\infty}^0 d\tau \delta((x - z(\tau))^2 + \epsilon^2) \frac{dL^{\lambda\mu}(\tau)}{d\tau}, \quad (\text{A11})$$

where the important quantity $L^{\lambda\mu}(\tau)$ is defined by:

$$L^{\lambda\mu}(\tau) \equiv \frac{(x - z(\tau))^\lambda v^\mu(\tau)}{(x - z(\tau)) \cdot v(\tau)}. \quad (\text{A12})$$

L satisfies the identity that its trace is unity:

$$L_\lambda^\lambda(\tau) \equiv \frac{(x - z(\tau)) \cdot v(\tau)}{(x - z(\tau)) \cdot v(\tau)} = 1, \quad (\text{A13})$$

which ensures that the regularized potential satisfies the Lorenz condition:

$$\partial_\lambda A_\epsilon^\lambda(x) = \frac{e}{2\pi} \int_{-\infty}^0 d\tau \delta((x - z(\tau))^2 + \epsilon^2) \frac{d}{d\tau} (1) = 0. \quad (\text{A14})$$

Defining $N^{\lambda\mu}(\tau) \equiv dL^{\lambda\mu}(\tau)/d\tau$, and using Eq. (A5) in Eq. (A11), the integration picks up only the contribution from $\tau = -\epsilon$, giving:

$$\partial^\lambda A_\epsilon^\mu(x) = +\frac{e}{4\pi} \frac{N^{\lambda\mu}(-\epsilon)}{\epsilon}. \quad (\text{A15})$$

From the results in Appendix B, we have the expansion:

$$N^{\lambda\mu}(-\epsilon) = -\left(v_0^\lambda a_0^\mu + \frac{1}{2} a_0^\lambda v_0^\mu\right) \quad (\text{A16a})$$

$$+ \epsilon \left(v_0^\lambda \dot{a}_0^\mu + a_0^\lambda a_0^\mu + \frac{\dot{a}_0^\lambda v_0^\mu - a^2 v_0^\lambda v_0^\mu}{3}\right) + O(\epsilon^2), \quad (\text{A16b})$$

with which we can perform two consistency checks on our computation. First, we can directly check that the Lorenz condition is maintained. From Eq. (A16a, A16b) (dropping the subscripts):

$$N_\lambda^\lambda(-\epsilon) = -(3/2)(v \cdot a) + \epsilon \left(v \cdot \dot{a} + a^2 + \frac{\dot{a} \cdot v + a^2}{3}\right) + O(\epsilon^2) \quad (\text{A17a})$$

$$= 0 + O(\epsilon^2) \quad (\text{A17b})$$

using the identities $v^2 = -1$, $a \cdot v = 0$, and $\dot{a} \cdot v + a^2 = 0$. A more subtle consistency check comes from applying $d/d\tau$ to Eq. (A8) to obtain:

$$\frac{dA^\mu(z)}{d\tau} = \frac{e}{4\pi} \left(\frac{a^\mu(\tau)}{\epsilon} - \dot{a}^\mu(\tau) \right) + O(\epsilon), \quad (\text{A18a})$$

$$= \partial^\lambda A^\mu v_\lambda, \quad (\text{A18b})$$

where the latter equation results from a simple application of the chain rule of differentiation. We can check directly that these two forms of the equation are consistent using Eq. (A16a, A16b):

$$\frac{N^{\lambda\mu}(-\epsilon)v_\lambda}{\epsilon} = -\frac{1}{\epsilon} \left(v^2 a^\mu + \frac{1}{2} v \cdot a v^\mu \right) + \left(v^2 \dot{a}^\mu + v \cdot a \dot{a}^\mu + \frac{v \cdot \dot{a} v^\mu - a^2 v^2 v^\mu}{3} \right) \quad (\text{A19a})$$

$$= \frac{a^\mu(\tau)}{\epsilon} - \dot{a}^\mu(\tau), \quad (\text{A19b})$$

where it should be noted that these consistency checks involve both divergent and nondivergent terms in the regularized expressions.

The field strength is given by the antisymmetric combination:

$$F_\epsilon^{\lambda\mu}(x) = +\frac{e}{4\pi} \frac{N^{\lambda\mu}(-\epsilon) - N^{\mu\lambda}(-\epsilon)}{\epsilon} \equiv \frac{e}{4\pi} \frac{N^{[\lambda\mu]}(-\epsilon)}{\epsilon}, \quad (\text{A20})$$

into which we substitute the results from Appendix B, resulting in the regularized field self field up to $O(\epsilon)$:

$$F_\epsilon^{\lambda\mu}(z(\tau)) = -\frac{1}{2\epsilon} \left(\frac{e}{4\pi} \right) (v^\lambda a^\mu - v^\mu a^\lambda) \quad (\text{A21a})$$

$$+ \frac{2}{3} \left(\frac{e}{4\pi} \right) (v^\lambda \dot{a}^\mu - v^\mu \dot{a}^\lambda) \quad (\text{A21b})$$

$$- \frac{3\epsilon}{8} \left(\frac{e}{4\pi} \right) \left[(v^\lambda \ddot{a}^\mu - v^\mu \ddot{a}^\lambda) + \frac{2}{3} (a^\lambda \dot{a}^\mu - a^\mu \dot{a}^\lambda) + \frac{2v \cdot \dot{a}}{3} (v^\lambda a^\mu - v^\mu a^\lambda) \right]. \quad (\text{A21c})$$

This explicit computation justifies the constants given in Eqs. (16, 17).

Appendix B: Expansion of $L^{\lambda\mu}$, $N^{\lambda\mu}$

Our objective is to expand Eq. (A12), rewritten here for convenience ($x = z(0)$):

$$L^{\lambda\mu}(\tau) \equiv \frac{(z(\tau) - x)^\lambda v^\mu(\tau)}{(z(\tau) - x) \cdot v(\tau)}, \quad (\text{B1})$$

using the Taylor expansions (dropping the Lorentz index):

$$z(\tau) = x + \tau v_0 + \frac{\tau^2}{2!} a_0 + \frac{\tau^3}{3!} \dot{a}_0 + \frac{\tau^4}{4!} \ddot{a}_0 + O(\tau^5), \quad (\text{B2a})$$

$$v(\tau) = v_0 + \tau a_0 + \frac{\tau^2}{2!} \dot{a}_0 + \frac{\tau^3}{3!} \ddot{a}_0 + O(\tau^4). \quad (\text{B2b})$$

The numerator in $L^{\lambda\mu}$ becomes:

$$(z(\tau) - x)^\lambda v^\mu(\tau) = \tau A^{\lambda\mu} + \tau^2 B^{\lambda\mu} + \tau^3 C^{\lambda\mu} + \tau^4 D^{\lambda\mu} + O(\tau^5), \quad (\text{B3})$$

where:

$$A^{\lambda\mu} = v_0^\lambda v_0^\mu, \quad (\text{B4a})$$

$$B^{\lambda\mu} = v_0^\lambda a_0^\mu + \frac{1}{2} a_0^\lambda v_0^\mu, \quad (\text{B4b})$$

$$C^{\lambda\mu} = \frac{1}{2} v_0^\lambda \dot{a}_0^\mu + \frac{1}{2} a_0^\lambda \dot{a}_0^\mu + \frac{1}{6} \dot{a}_0^\lambda v_0^\mu, \quad (\text{B4c})$$

$$D^{\lambda\mu} = \frac{1}{6} v_0^\lambda \ddot{a}_0^\mu + \frac{1}{4} a_0^\lambda \ddot{a}_0^\mu + \frac{1}{6} \dot{a}_0^\lambda a_0^\mu + \frac{1}{24} \ddot{a}_0^\lambda v_0^\mu. \quad (\text{B4d})$$

Contracting yields the expansion for the denominator:

$$(z(\tau) - x) \cdot v(\tau) = -\tau + \tau^3 \frac{v_0 \cdot \dot{a}_0}{6} - \tau^4 \frac{5\dot{a}_0^2}{24} + O(\tau^5), \quad (\text{B5})$$

where the identities $v^2 = -1$, $v \cdot a = 0$, $a^2 + v \cdot \dot{a} = 0$, and $2a \cdot \dot{a} + \dot{a}^2 + v \cdot \ddot{a} = 0$ have been employed. With these expansions, L becomes:

$$L^{\lambda\mu}(\tau) = \frac{-1}{1 - \tau^2 \frac{v_0 \cdot \dot{a}_0}{6} + \tau^3 \frac{5\dot{a}_0^2}{24} + O(\tau^4)} (A^{\lambda\mu} + \tau B^{\lambda\mu} + \tau^2 C^{\lambda\mu} + \tau^3 D^{\lambda\mu} + O(\tau^4)), \quad (\text{B6a})$$

$$= - \left(1 + \tau^2 \frac{v_0 \cdot \dot{a}_0}{6} - \tau^3 \frac{5\dot{a}_0^2}{24} + O(\tau^4) \right) (A^{\lambda\mu} + \tau B^{\lambda\mu} + \tau^2 C^{\lambda\mu} + \tau^3 D^{\lambda\mu} + O(\tau^4)), \quad (\text{B6b})$$

$$= -A^{\lambda\mu} - \tau B^{\lambda\mu} - \tau^2 \left(C^{\lambda\mu} + \frac{v_0 \cdot \dot{a}_0}{6} A^{\lambda\mu} \right) - \tau^3 \left(D^{\lambda\mu} + \frac{v_0 \cdot \dot{a}_0}{6} B^{\lambda\mu} - \frac{5\dot{a}_0^2}{24} A^{\lambda\mu} \right) + O(\tau^4), \quad (\text{B6c})$$

which gives $N^{\lambda\mu} = dL^{\lambda\mu}/d\tau$ as:

$$N^{\lambda\mu} = -B^{\lambda\mu} - 2\tau \left(C^{\lambda\mu} + \frac{v_0 \cdot \dot{a}_0}{6} A^{\lambda\mu} \right) - 3\tau^2 \left(D^{\lambda\mu} + \frac{v_0 \cdot \dot{a}_0}{6} B^{\lambda\mu} - \frac{5\dot{a}_0^2}{24} A^{\lambda\mu} \right) + O(\tau^4). \quad (\text{B7})$$

Inserting $\tau = -\epsilon$ into this equation, and dropping the “0” subscripts, gives:

$$N^{\lambda\mu}(-\epsilon) = - \left(v^\lambda a^\mu + \frac{a^\lambda v^\mu}{2} \right) \quad (\text{B8a})$$

$$+ \epsilon \left[v^\lambda \dot{a}^\mu + a^\lambda \dot{a}^\mu + \frac{1}{3} (\dot{a}^\lambda v^\mu - a^2 v^\lambda v^\mu) \right] \quad (\text{B8b})$$

$$- 3\epsilon^2 \left[\frac{1}{6} v^\lambda \ddot{a}^\mu + \frac{1}{4} a^\lambda \ddot{a}^\mu + \frac{1}{6} \dot{a}^\lambda a^\mu + \frac{1}{24} \ddot{a}^\lambda v^\mu + \frac{v \cdot \dot{a}}{6} \left(v^\lambda a^\mu + \frac{a^\lambda v^\mu}{2} \right) - \frac{5\dot{a}^2}{24} v^\lambda v^\mu \right], \quad (\text{B8c})$$

which is used in the derivation of the self-field in Appendix A. It is a useful exercise to contract this expression and demonstrate that it vanishes to the order indicated. The last $O(\epsilon^2)$ term, which comes from the expansion of the denominator in L , is crucial to obtaining this result. For completeness, I give the antisymmetric sum, which is used in the derivation of the self-field:

$$N^{[\lambda\mu]}(-\epsilon) = -\frac{1}{2} (v^\lambda a^\mu - v^\mu a^\lambda) \quad (\text{B9a})$$

$$+ \frac{2\epsilon}{3} [v^\lambda \dot{a}^\mu - v^\mu \dot{a}^\lambda] \quad (\text{B9b})$$

$$- 3\epsilon^2 \left[\frac{1}{8} (v^\lambda \ddot{a}^\mu - v^\mu \ddot{a}^\lambda) + \frac{1}{12} (a^\lambda \dot{a}^\mu - a^\mu \dot{a}^\lambda) + \frac{v \cdot \dot{a}}{12} (v^\lambda a^\mu - v^\mu a^\lambda) \right]. \quad (\text{B9c})$$

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² H.A. Lorentz, *The Theory of Electrons and Its Applications to the Phenomena of Light and Radiant Heat*, 2nd ed. (B.G. Teubner, Leipzig, 1916). See Section 26ff.

- ³ Julian Schwinger, ed., *Selected Papers on Quantum Electrodynamics* (Dover, New York, 1958).
- ⁴ A succinct summary, with references, is in Chapter 1 of Steven Weinberg, *The Quantum Theory of Fields, Volume 1* (Cambridge Univ. Press, Cambridge, 1995).
- ⁵ F. Rohrlich, *Classical Charged Particles: Foundations of their Theory* (Addison-Wesley, Reading, MA 1965). One of those exceptions is P.A.M. Dirac, “Classical theory of radiating electrons,” *Proc. Roy. Soc. Lond.* **A167**, 148–169 (1938).
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- ⁷ This was clearly Rohrlich’s point of view in Ref. 5 (pp. 128–129): “Other methods go under the name ‘cutoff procedure’ and ‘renormalization technique.’ . . . These methods provide no solution to the problem. Their accomplishment lies solely in arriving at those physically meaningful structure-independent results which would have been obtained had the theory been formulated in a structure-independent way from the very beginning. . . they are entirely unsatisfactory from the point of view of physical theory.”
- ⁸ S.E. Gralla, A.I. Harte, and R.M. Wald, “Rigorous derivation of electromagnetic self-force,” *Phys. Rev.* **D80**, 024031 (2009).
- ⁹ The point of view taken in this paper is closest to G.P. Lepage’s “What is renormalization?” [hep-ph/0506330](#) and “How to renormalize the Schrödinger equation,” [nucl-th/9706029](#). Other useful perspectives are given in David B. Kaplan, “Effective field theories,” [nucl-th/9506035](#), and in Barry Holstein, “Blue skies and effective interactions,” *Am. J. Phys.* **67**, 422–427 (1999), and his “Effective interactions are effective interactions,” [hep-ph/0010033](#).
- ¹⁰ The equations can be found in Ref. 5. We use the metric $g^{\mu\nu} = \text{diag}(- + + +)$ and set $c = 1 = \mu_0 = \epsilon_0^{-1}$. The particle coordinate four-vector is $z^\mu(\tau)$, where τ is the proper time. An overdot indicates differentiation with respect to the proper time: $\dot{z}^\mu \equiv dz^\mu/d\tau \equiv v^\mu$, $\ddot{z}^\mu \equiv a^\mu$, etc. We will often abbreviate inner products: $a^\lambda b_\lambda \equiv a \cdot b$, $a^\lambda a_\lambda \equiv a^2$, etc.
- ¹¹ K.G. Wilson, “The renormalization group and critical phenomena,” *Rev. Mod. Phys.* **55**, 583 (1983).
- ¹² A useful analogy is to the expansion of a static charge distribution in multipole moments and keeping the first few terms.
- ¹³ The v could also be contracted with an a , but this vanishes because $v \cdot a = 0$.

- ¹⁴ The $O(\epsilon)$ constants are given in Appendix A: $4\pi c_1^{(1)} = -3/8$, $4\pi c_1^{(2)} = -1/4 = 4\pi c_1^{(3)}$ using the delta regulator employed in this paper.
- ¹⁵ The “ τ_0^2 ” factor in the “O” sign comes from the fact that each higher order contribution to the self-field carries with it an extra factor of e_{phys}^2 which, on division by m_{phys} , provides an extra factor of τ_0 that combines with the explicit factor of $\epsilon \sim \tau_0$.
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